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Distribution of Controlled Lyapunov Exponents via the Lai-Chen Algorithm

D. Lai
Division of Biostatistics
School of Public Health
University of Texas, Houston, TX 77030, U.S.A.
and
Faculty of Statistics
Jiangxi University of Finance and Economics
Nanchang, Jiangxi, P.R. China

G. Chen
Dept. of Electronic Engineering
City University of Hong Kong
Kowloon, Hong Kong SAR, P.R. China

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Abstract—The Lai-Chen algorithm, an extended feedback control scheme of the original Chen-Lai algorithm, was proposed to gradually make an arbitrarily given discrete-time dynamical system chaotic in terms of possessing positive Lyapunov exponents with uniformly bounded orbits. In this paper, based on the Monte Carlo method, we further study the distribution of the controlled Lyapunov exponents generated by the Lai-Chen algorithm. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Anticontrol, Chaos, Dynamical systems, Lyapunov exponent, Monte Carlo simulation, Time series.

1. INTRODUCTION

Since the introduction of the Chen-Lai algorithm in chaotifying nonchaotic discrete dynamical systems [1], there have been many studies on the mathematical properties of the algorithm based on various definitions of chaos [2–4]. Recently, the Chen-Lai algorithm, with its original feedback technique, was relaxed to a gradually control scheme that may be more convenient in some applications where a rigid control is difficult to implement [5]. To name just a couple of beneficial applications of the algorithms, it is noted that preserving chaos may provide a constructive role in controlling human health [6,7].

In this paper, following the similar study of the statistical behaviour of the controlled Lyapunov exponents in [8] on the Chen-Lai algorithm, we studied the empirical distribution of the controlled Lyapunov exponents generated under the Lai-Chen algorithm based on Monte Carlo simulations.

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The Monte Carlo method works well in generating the distributions of uncontrolled dynamical systems [9] and the controlled systems [8]. To summarize the controlled Lyapunov exponents resulting from the Lai-Chen algorithm, we fitted against the Gamma density functions to the observed Lyapunov exponents simulated in our experiments.

For easy reference, we first review the Lai-Chen algorithm in Section 2. The simulation results and the fitted Gamma density function are then given in Section 3. Some concluding remarks are finally presented in Section 4.

2. THE LAI-CHEN ALGORITHM

Consider an initially nonchaotic discrete-time nonlinear dynamical system of the form

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k), \quad \mathbf{x}_k \in R^n, \\ \mathbf{x}_0 - \text{given}, \end{cases} \tag{1}$$

where $\mathbf{f}_k$ is assumed to be continuously differentiable, at least locally in a region of interest, as defined in [1]. Our objective is to design a control input sequence, $\{\mathbf{u}_k\}_{k=0}^{\infty}$, such that the outputs (or, state vectors) of the controlled system

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{u}_k, \\ \mathbf{x}_0 - \text{given} \end{cases} \tag{2}$$

behave chaotically, in the sense that all the Lyapunov exponents of this controlled system become positive eventually while all system orbits are confined within a uniformly bounded region. [5].

In achieving this goal of "anticontrol of chaos" (or, "chaotification") for dynamical system (1), we would find a simple control sequence, such as a linear state-feedback control sequence, having the standard structure of

$$\mathbf{u}_k = B_k \mathbf{x}_k, \tag{3}$$

where $\{B_k\}$ are $n \times n$ constant matrices to be determined. We will not try to tune any of the system parameters since oftentimes it is difficult or even impossible. Thus, the controlled system (2) becomes

$$\begin{cases} \mathbf{x}_{k+1} = \tilde{\mathbf{f}}_k(\mathbf{x}_k) := \mathbf{f}_k(\mathbf{x}_k) + B_k \mathbf{x}_k, (\text{mod } 1) \\ \mathbf{x}_0 - \text{given}, \end{cases} \tag{4}$$

where the "mod" operation guarantees that the controlled system orbits are uniformly bounded.

Next, let

$$J_j(\mathbf{z}) := \tilde{\mathbf{f}}_j'(\mathbf{z}) = \mathbf{f}_j'(\mathbf{z}) + B_j \tag{5}$$

be the Jacobian of $\tilde{\mathbf{f}}_j(\cdot)$, evaluated at $\mathbf{z}$, $j = 0, 1, 2, \ldots$, and let

$$T_j = T_j(\mathbf{x}_0, \ldots, \mathbf{x}_j) := J_j(\mathbf{x}_j) J_{j-1}(\mathbf{x}_{j-1}) \ldots J_1(\mathbf{x}_1) J_0(\mathbf{x}_0). \tag{6}$$

Moreover, let $\mu_i^j = \mu_i(T_j^\top T_j)$ be the i^{th} eigenvalue of the j^{th} product matrix $[T_j^\top T_j]$, where $i = 1, \ldots, n$ and $j = 0, 1, 2, \ldots$. As is well known (see [10] or [11]), the i^{th} Lyapunov exponent of the orbit $\{\mathbf{x}_k\}_{k=0}^{\infty}$ of the controlled system (2), starting from the given $\mathbf{x}_0$, is defined by

$$\begin{aligned} \lambda_i(\mathbf{x}_0) &= \lim_{k \to \infty} \frac{1}{2k} \ell n \left| \mu_i \left(T_k^\top T_k \right) \right| \\ &= \lim_{k \to \infty} \frac{1}{2k} \ell n \left| \mu_i \left(J_0^\top(\mathbf{x}_0) \ldots J_k^\top(\mathbf{x}_k) J_k(\mathbf{x}_k) \ldots J_0(\mathbf{x}_0) \right) \right|, \end{aligned} \tag{7}$$

where $i = 1, \ldots, n$.

In the controlled system (4), under some conditions to be described below, we can design the constant matrices $\{B_k\}_{k=0}^{\infty}$ such that all the Lyapunov exponents of the system orbit $\{\mathbf{x}_k\}_{k=0}^{\infty}$ are positive and finite eventually [5]

$$0 < c \leq \lambda_i(\mathbf{x}_0) < \infty, \qquad i = 1, \ldots, n, \tag{8}$$

where c is a pre-desired constant.

At the initial step, $k = 0$, we determine the control-gain matrix B_0 such that $[T_0 T_0^\top]^{-1} > 0$. Then, for each $k = 1, 2, \ldots$, we determine the control-gain matrix B_k such that

(i) $[T_{k-1}T_{k-1}^\top]^{-1} > 0$, and
(ii) $[J_k^\top J_k] - e^{2kc_k}\, [T_{k-1}T_{k-1}^\top]^{-1} \geq 0$,

where, for the given constant $c > 0$, we let $c_k \to c$ as $k \to \infty$. We use $A > 0$ (resp., $A \geq 0$) to denote a finite and positive definite (resp., nonnegative definite) matrix A. The desired control sequence can be obtained by the following algorithm.

Start with the feedback controlled system $\mathbf{x}_1 = \mathbf{f}_0(\mathbf{x}_0) + B_0\mathbf{x}_0$, where $\mathbf{x}_0$ is initially given. Calculate its Jacobian $J_0(\mathbf{x}_0) = \mathbf{f}_0'(\mathbf{x}_0) + B_0$ and let $T_0 = J_0(\mathbf{x}_0)$. Design a positive feedback control gain $B_0 = \sigma_0 I_n$ by choosing a positive number $\sigma_0 > 0$ such that the matrix $\left[T_0 T_0^\top\right]$ is finite and diagonally dominant.

For $k = 0, 1, 2, \ldots$, start with the controlled system $\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + B_k\mathbf{x}_k$, where both $B_k = \sigma_k I_n > 0$ and $\mathbf{x}_k$ were obtained from the previous step:

1. compute the Jacobian $J_k(\mathbf{x}_k) = \mathbf{f}_k'(\mathbf{x}_k) + \sigma_k I_n$ and then let $T_k = J_k T_{k-1}$;
2. design a positive feedback controller by choosing the positive number σ_k such that the matrix $[T_k T_k^{-1}] - e^{2kc_k} I_n$ is finite and diagonally dominant, where $c_k c$ and the constant $c > 0$ is the one given in (8).

The control-gain sequence $\{B_k\} = \{\sigma_k I_n\}$ is chosen such that at step k, σ_k satisfies

$$\begin{aligned} &J_k^\top J_k - e^{2kc_k}\left[T_{k-1}T_{k-1}^\top\right]^{-1} \\ &= [\mathbf{f}_k'(\mathbf{x}_k)]^\top [\mathbf{f}_k'(\mathbf{x}_k)] + \sigma_k\left([\mathbf{f}_k'(\mathbf{x}_k)]^\top + [\mathbf{f}_k'(\mathbf{x}_k)]\right) + \sigma_k^2 I_n - e^{2kc_k}\left[T_{k-1}T_{k-1}^\top\right]^{-1} \geq 0. \end{aligned} \tag{9}$$

This can be achieved if we let the matrix in (9) be diagonal dominant by a properly chosen real number σ_k. A mathematical justification of the algorithm was given in [5].

Similarly to [8], we know that the selection of σ_k in the algorithm is not unique. There are infinitely many ways to select the control-gain sequence that satisfies the nonnegative definite condition (9), therefore σ_k can be any value in a suitable region. For example, σ_k can be chosen uniformly as in [5].

In this paper, we investigate the distribution of the resulting positive Lyapunov exponents in correspondence to the different choices of the control-gain sequences.

3. SIMULATION RESULTS

The analytic distribution of the controlled Lyapunov exponent is very difficult, if not impossible, to derive. We therefore resort to its numerical distribution using the Monte Carlo method. It turns out that simulation is an effective way of examining the distribution of the controlled Lyapunov exponents. In our experiments, we generated the control-gain sequence uniformly in the given intervals and then computed the controlled Lyapunov exponents and produced histograms of the controlled Lyapunov exponents. The family of Gamma density functions were fitted to the simulated Lyapunov exponents.

As in [8], we continued our study on two typical feedback anticontrolled dynamical systems under the Lai-Chen algorithm. The first one was the logistic map, originally not in a chaotic state:

$$x_{k+1} = 3x_k(1 - x_k), \qquad \bmod (2.5).$$

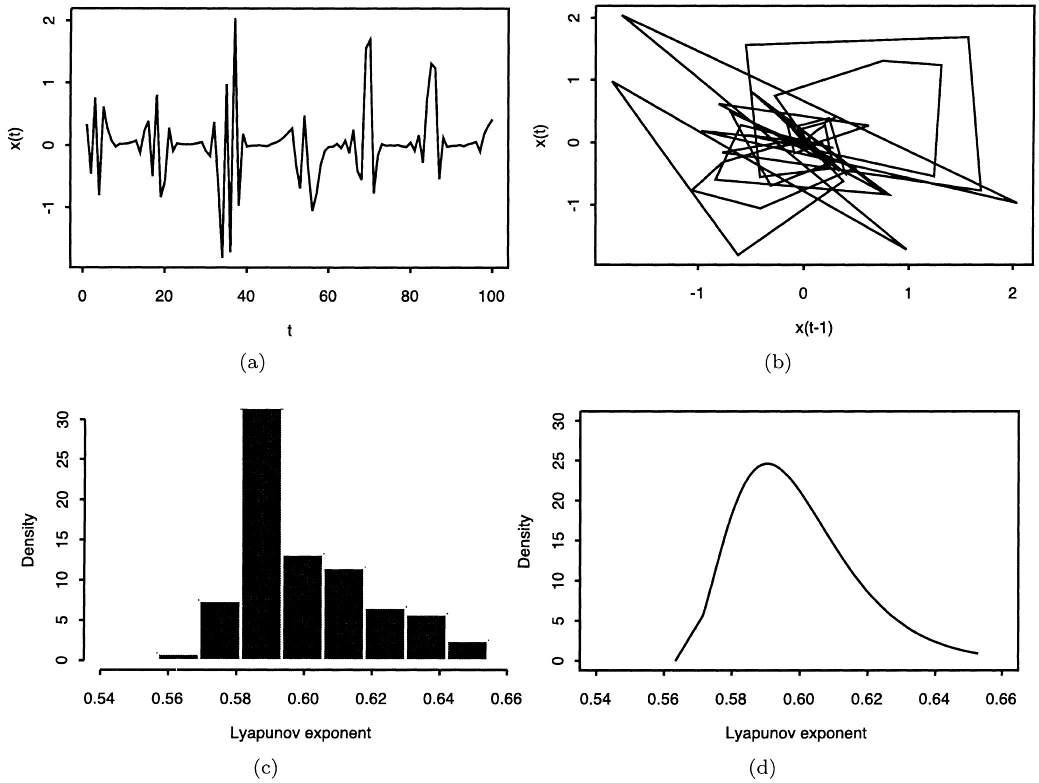


Figure 1. Results of anticontrol simulation for $x_{t+1}=3x_t(1-x_t)$ (mod) 2.5 with duration $1 \le t \le 100$: (a) a typical realization; (b) the phase-space plot of (a); (c) the simulated Lyapunov exponents; (d) the fitted Gamma density function of (c).

To achieve the goal of controlling the Lyapunov exponents so that eventually $c = 0.3$, we let the control gains be $c_k = c + 1/k$, $k = 1, 2, \ldots$. Hence, $c_k \to c$ as $k \to \infty$. The sequence $\{\sigma_k\}$ are picked randomly and uniformly from the intervals (determined in [5]):

$$\left[-f'(x_k) - e^{kc_k}(T_{k-1})^{-1} - 0.01,\ -f'(x_k) - e^{kc_k}(T_{k-1})^{-1}\right]$$

and

$$\left[-f'(x_k) + e^{kc_k}(T_{k-1})^{-1},\ -f'(x_k) + e^{kc_k}(T_{k-1})^{-1} + 0.01\right].$$

The control gain sequence $\{\sigma_k\}$ picked from these two intervals satisfy condition (9) that makes the Lyapunov exponents gradually become larger than $c = 0.3$.

A typical realization of the series with the simulation duration $0 \le k \le 100$ and its phase-space plot are shown in Figures 1a and 1b, respectively. The Lyapunov exponents of the anticontrolled system orbits were then computed. The histogram of 100 realizations of the controlled Lyapunov exponents from 100 simulated time series is shown in Figure 1c. The mean and the variance of these Lyapunov exponents are found to be 0.5995 and 3.3301×10^{-4}, respectively. The Gamma density functions appear in various shapes. To empirically assess the distribution of the controlled Lyapunov exponents, we chose the two-parameter Gamma family of densities in the form of

$$x^{\alpha-1}e^{-x/\beta}/(\Gamma(\alpha)\beta^{\alpha}), \qquad x > 0.$$

To fit the density function to the controlled Lyapunov exponents, we first transformed the observed estimates of the Lyapunov exponents by subtracting the minimum value (0.5634) of the observations from each of the controlled Lyapunov exponents, and then let the variable x (in the Gamma density) be the transformed Lyapunov exponent. The estimation procedure is based on

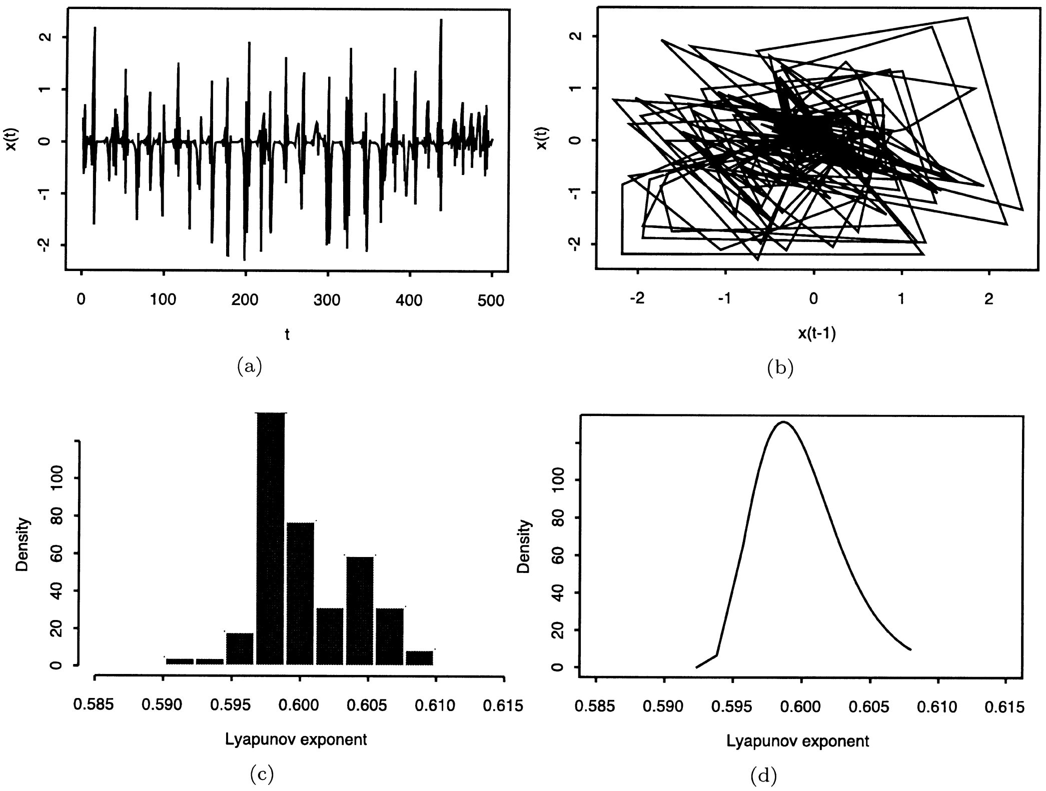


Figure 2. Results of anticontrol simulation for $x_{t+1}{=}3x_t(1 - x_t)$ (mod) 2.5 with duration $1 \le t \le 500$: (a) a typical realization; (b) the phase-space plot of (a); (c) the simulated Lyapunov exponents; (d) the fitted Gamma density function of (c).

the method of moments. The estimates of the two parameters α and β in the Gamma density, for the 100 realizations of the transformed Lyapunov exponents based on the time series with length 100, were 3.9219 and 0.0092, respectively. The estimated Gamma density function is plotted in Figure 1d.

The second simulation on the same logistic system was similar, except that the simulation duration was longer: $0 \le k \le 500$. A typical realization of the series with simulation duration: $0 \le k \le 500$ is shown in Figure 2a. The histogram of the estimates of the Lyapunov exponent, from 100 simulated time series with duration 500, is shown in Figure 2c. The mean and the variance of the Lyapunov exponents are 0.6000 and 1.0826×10^{-5}, respectively. The same method was used for density estimation of the transformed Lyapunov exponents. The estimates of the two parameters α and β at this time were 5.4564 and 0.0014, respectively. The plot of the estimated density function of the Lyapunov exponents is shown in Figure 2d.

The second example of an uncontrolled system is the following simple linear map:

$$x_{k+1} = -(1/2)x_k, \qquad \bmod (2.5).$$

Clearly, this linear system is asymptotically stable at the equilibrium 0, so it is by no mean chaotic. To make it behave chaotically, we chose $c = 0.03$ and $c_k = c + 1/k$, $k = 1, 2, \dots,$ and let the control gain sequence $\{\sigma_k\}$ be picked randomly and uniformly from the following two intervals (determined in [5]):

$$\left[-f'(x_k) - e^{kc_k}(T_{k-1})^{-1} - 0.05,\ -f'(x_k) - e^{kc_k}(T_{k-1})^{-1}\right]$$

and

$$\left[-f'(x_k) + e^{kc_k}(T_{k-1})^{-1},\ -f'(x_k) + e^{kc_k}(T_{k-1})^{-1} + 0.05\right].$$

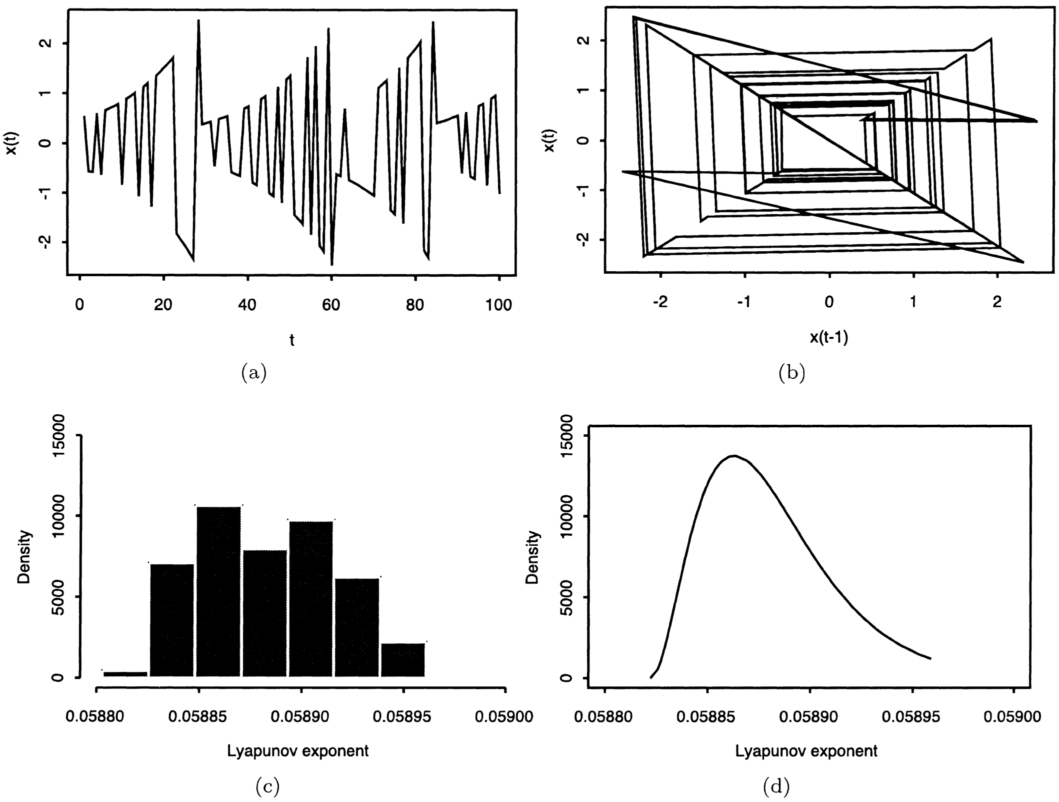


Figure 3. Results of anticontrol simulation for $x_{t+1}{=}-1/2x_t$ (mod) 2.5 with duration $1 \leq t \leq 100$: (a) a typical realization; (b) the phase-space plot of (a); (c) the simulated Lyapunov exponents; (d) the fitted Gamma density function of (c).

The simulation duration was $0 \leq k \leq 100$ in the first simulation. A typical realization of the series is shown in Figure 3a. The Lyapunov exponents of the anticontrolled system orbits were then calculated. The histogram of 100 realizations of the Lyapunov exponents from 100 simulations of the simulated time series is presented in Figure 3c. The mean and the variance of the Lyapunov exponents are 0.0589 and 1.1488×10^{-9}, respectively. The estimated parameters of the Gamma density function were $\alpha = 3.1030$ and $\beta = 1.9241^{-5}$, respectively. Figure 3d gives the plot of the estimated density function.

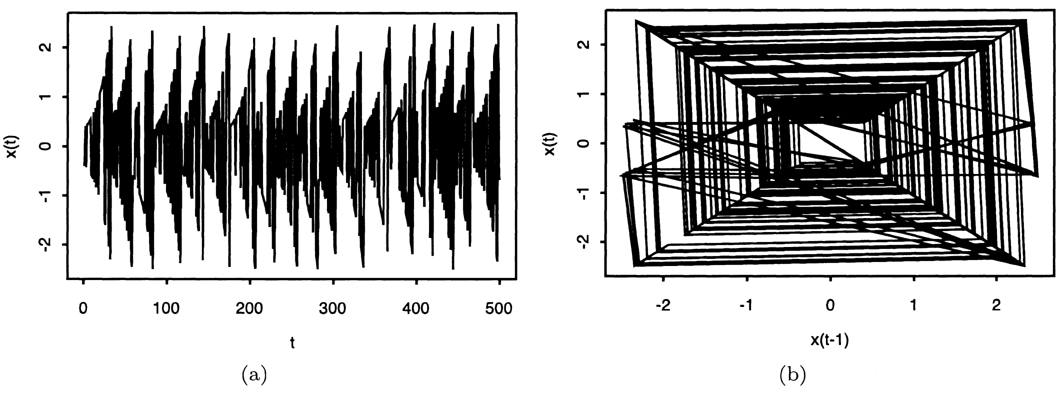
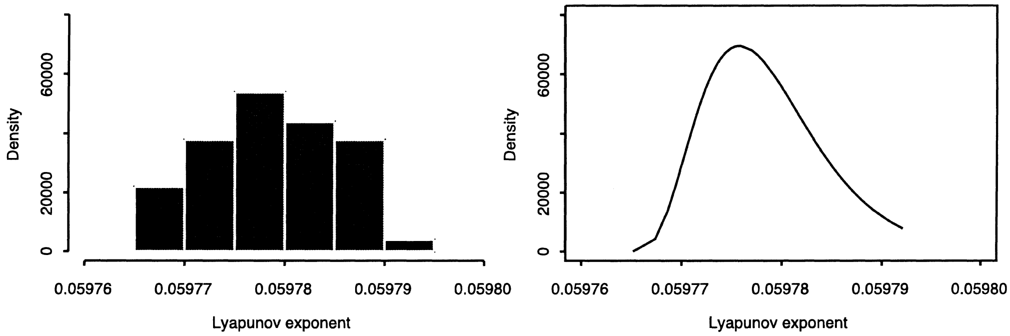


Figure 4. Results of anticontrol simulation for $x_{t+1}{=}-1/2x_t$ (mod) 2.5 with duration $1 \leq t \leq 500$: (a) a typical realization; (b) the phase-space plot of (a); (c) the simulated Lyapunov exponents; (d) the fitted Gamma density function of (c).



(c) (d)

Figure 4. (cont.)

The second simulation of this example was basically the same, except that the simulation duration was longer: $0 \leq k \leq 500$. A typical realization of the series is shown in Figure 4a. The histogram of the estimates of the controlled Lyapunov exponents, using 100 simulation time series of length 500, is shown in Figure 4c. The mean and variance of the controlled Lyapunov exponents are 0.0598 and 4.0231×10^{-11}, respectively. The estimated parameters α and β in this case were 4.5714 and 2.9666×10^{-6}, respectively. The plot of the estimated density function is shown in Figure 4d.

4. CONCLUDING REMARKS

Chen and Lai [1] proposed a simple yet rigorous feedback anticontrol design. The algorithm was lately extended in [5], which can make an originally nonchaotic system chaotic in the sense that all the controlled Lyapunov exponents are strictly positive gradually while the system orbits are uniformly bounded. We noticed that the choice of an anticontrol gain sequence is not unique, therefore some random mechanisms can be used to suitably select them. With this study, understanding the behavior of the controlled Lyapunov exponent is important for further statistical inference.

In this paper, we have examined the distribution of such controlled Lyapunov exponents via the Monte Carlo simulation. As in [8], we have observed, from the simulation results under the Lai-Chen algorithm, that the distribution of the controlled Lyapunov exponents had a right tail with a peak on the left. A Gamma density function seemed very reasonable for modelling this type of data. In this paper, therefore, we continued to use the Gamma density function on empirical data fitting for the simulated Lyapunov exponents under the Lai-Chen algorithm. Our simulation studies have shown that the Gamma density functions indeed work quite well for the purpose.

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